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On the conformal higher spin unfolded equation for a three-dimensional self-interacting scalar field

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Abstract: We propose field equations for the conformal higher spin system in three dimensions coupled to a conformal scalar field with a sixth order potential. Both the higher spin equation and the unfolded equation for the scalar field have source terms and are based on a conformal higher spin algebra which we treat as an expansion in multi-commutators. Explicit expressions for the source terms are suggested and subjected to some simple tests. We also discuss a cascading relation between the Chern-Simons action for the higher spin gauge theory and an action containing a term for each spin that generalizes the spin 2 Chern-Simons action in terms of the spin connection expressed in terms of the frame field. This cascading property is demonstrated in the free theory for spin 3 but should work also in the complete higher spin theory.

Keywords: Chern-Simons theory, higher spins, AdS/CFT

Contents

1 Introduction

The purpose of this paper is to propose a set of field equations in three dimensions that describe a fully interacting conformal system consisting of a scalar field and the higher spin theory generated by the $SO(3,2)$ higher spin algebra. We will follow the work [1] where this approach to the problem was discussed in some detail and some results were found indicating that this may be worth pursuing further.

After giving the proposed equations in section 2 we will first explain the notation and content of them and then present some of the arguments leading to their particular form together with some explicit checks that will give some support for this proposal. The higher spin part of the theory has its origin in a gauged $SO(3,2)$ Chern-Simons theory which can be reformulated as a generalization to all higher spins of the standard spin 2 Chern-Simons theory for the spin connection. This will be elaborated upon in section 3 where a cascading trick is used to relate the two different Chern-Simons formulations of the higher spin theory. Some additional comments are collected in the Conclusions.

Thus our main goal will be to present two higher spin equations, one field strength and one unfolded equation, and to show that the following spin 0 (Klein-Gordon) and spin 2 (Cotton) equations can be reproduced:

$$\Box\phi - \frac{1}{8}R\phi - \frac{27g^3}{32\cdot 32}\phi^5 = 0, \tag{1.1}$$

and

$$\begin{aligned} &C_{\mu\nu} - \frac{g}{16}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\phi^2 - \frac{g}{2}(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial^\rho\phi\partial_\rho\phi) \\ &-\frac{g}{8}(\phi\Box\phi + \partial^\rho\phi\partial_\rho\phi)g_{\mu\nu} + \frac{g}{8}(\phi D_\mu\partial_\nu\phi + \partial_\mu\phi\partial_\nu\phi) + \frac{9g^3}{2\cdot 32\cdot 32}\phi^6 = 0. \end{aligned} \tag{1.2}$$

Note, however, that these equations are taken from the topological gauging of three dimensional CFTs with eight supersymmetries [2, 3] where all coupling constants are determined in terms of the gravitational one g. This needs not to be the case in non-supersymmetric theories like the ones we deal with here.

Once the higher spin equations are presented we can discuss their consequences for the field equations for spin 3 and above. Only a few such comments will be given here while a more extensive survey will be left for a future publication. We may note already at this point that the spin 2 equation above will be augmented by new terms with more than two derivatives of the scalar $\phi(x)$ provided higher spin frame fields also appear. This is true also for the equations of spin 3 etc and follows directly from the fact that ϕ has conformal dimension $L^{-\frac{1}{2}}$ and that the number of derivatives in the spin s equation is $2s-1$ which implies that the spin s frame field itself is of dimension L^{s-2}. Also the Klein-Gordon equation will contain terms with higher spin fields and more than two derivatives on the scalar.

We will find it convenient to write the Cotton equation (1.2) in irreps of $SO(1,2)$. The point is that the trace is exactly the Klein-Gordon equation which means that the rest of the Cotton equation is in the irrep **5** and reads

$$C_{\mu\nu} - \frac{g}{16}(\phi^2 R_{\mu\nu} - 2\phi D_\mu\partial_\nu\phi + 6\partial_\mu\phi\partial_\nu\phi)|_{\mathbf{5}} = 0, \tag{1.3}$$

where we recall that the Cotton tensor is already in this irrep. The purpose of this paper is to suggest two higher spin field equations containing component equations for all spins ≥ 2 coupled to a scalar field ϕ with ϕ^6 potential and which in particular reproduce both the above Klein-Gordon and spin 2 Cotton equations. Already in [1] where this approach was discussed, but without source terms in either equation, it was shown that the correct curvature scalar term does arise in the Klein-Gordon equation and in addition also a spin 3 contribution[1]

$$\Box\phi - \frac{1}{8}R\phi + \tilde{f}\phi = 0, \tag{1.4}$$

where the spin 3 term contains the trace $\tilde{f} := e^{\mu}{}_{a}\tilde{f}_{\mu}{}^{a}(1,3)$. The field $\tilde{f}_{\mu}{}^{a}(1,3)$, which is an expression containing three derivatives on the spin 3 frame field $e_{\mu}{}^{ab}$ is discussed briefly later in this paper. The reader is advised to consult [1] for definitions and more details on the spin 3 sector of the higher spin system. The problematic issue of constructing source terms was mentioned in this context at the end of that paper, and a suggestion how it can be solved is presented in the next section.

The higher spin algebras together with the linearized versions of the zero field strength and unfolded equations have been discussed in many papers in the past, see, e.g., [4–10] and referencies therein. The conformal higher spin sector of the theory that is the subject of this paper is also analyzed in a recent paper by Vasiliev [11] where its relation to higher spin theory in AdS_4 is used to draw conclusions about Lagrangians etc. The scalar sectors, on the other hand, are not the same. In fact, the scalar considered in this paper is the one discussed in [9]. The linearized spin 3 frame field system used in section 4 below is discussed in the "metric" formulation in, e.g., [12]. Furthermore, the explicit analysis of the conformal higher spin system performed in this paper is closely related to the more formal approach of σ_- cohomology developed by Shaynkman and Vasiliev, see, e.g., [13–15].

2 The conformal interacting higher spin equations

The two basic field equations for the $SO(3,2)$ conformal higher spin (HS) theory coupled to a scalar field with fifth order self-interactions that we propose and study here are the *unfolded* equation

$$\mathcal{D}\Phi|0\rangle_q = S|0\rangle_q, \tag{2.1}$$

where $\mathcal{D} = d + A$, and the following *field strength* equation valued in the $SO(3,2)$ higher spin algebra

$$F = T. \tag{2.2}$$

2.1 The higher spin setup

We now explain the notation and content of these equations following [1]. $F = dA + A \wedge A$ is the HS field strength obtained from the HS gauge field A with the expansion

$$A = \Sigma_{n=1}^{\infty}(-i)^n A_n, \quad A_n = e^{a_1....a_n}P_{a_1....a_n} + ... + f^{a_1....a_n}K_{a_1....a_n}. \tag{2.3}$$

[1] There are probably also spin 3 terms with one or more derivatives on the scalar field.

To understand the structure of this gauge field we give the parts of the higher spin conformal system that will explicitly play a role below, namely the spin 2 part

$$A_1 = e^a P_a + \omega^a M_a + bD + f^a K_a, \tag{2.4}$$

where the generators of translation, Lorentz, dilatation and special conformal transformations are, respectively, P^a, M^a, D and K^a with their associated gauge fields e^a, ω^a, b and f^a, and the spin 3 part

$$A_2 = e^{ab} P_{ab} + \tilde{e}^{ab} \tilde{P}_{ab} + \tilde{e}^a \tilde{P}_a + \tilde{\omega}^{ab} \tilde{M}_{ab} + \tilde{\omega}^b \tilde{M}_a + \tilde{b} \tilde{D} + \tilde{f}^a \tilde{K}_a + \tilde{f}^{ab} \tilde{K}_{ab} + f^{ab} K_{ab}. \tag{2.5}$$

The gauge fields (lower case quantities) and generators (upper case) of the HS algebra appearing in these expressions are all in irreps, i.e., the $a_1....a_n$ are totally symmetric and traceless sets of three-dimensional vector indices. The fields $e^{a_1....a_n}$ are the spin $s = n+1$ frame fields and we will call $f^{a_1....a_n}$ the Schouten tensor[2] since $Df^{a_1....a_n} + ... = 0$ turns out to be the spin $s = n+1$ Cotton equation which is of order $2s-1$ in derivatives. We emphasize here that all the fields depend only on the three dimensional space-time coordinates x^μ and there are thus no dependence on any other coordinates or auxiliary variables like the ones[3] often appearing in Vasiliev's constructions of interacting higher spin theories in AdS.

The HS algebra can be defined as follows. Consider the $so(2,1) \approx sp(2,\mathbf{R})$ spinor variables q^α, p_α (with $\alpha, \beta, .. = 1, 2$) which are hermitian operators satisfying $[q^\alpha, p_\beta] = i\delta^\alpha_\beta$. The spin $s = n+1$ HS generators are then given by all Weyl ordered polynomials in q^α, p_α of degree $2n$. For example, for $s = 2$ we have

$$P^a(2,0) = -\frac{1}{2}(\sigma^a)_{\alpha\beta} q^\alpha q^\beta, \;\; M^a(1,1) = -\frac{1}{2}(\sigma^a)_\alpha{}^\beta q^\alpha p_\beta, \tag{2.6}$$

$$D(1,1) = -\frac{1}{4}(q^\alpha p_\alpha + p_\alpha q^\alpha), \;\; K^a(0,2) = -\frac{1}{2}(\sigma^a)^{\alpha\beta} p_\alpha p_\beta \tag{2.7}$$

and for $s = 3$

$$P^{ab}(4,0) = \frac{1}{4}(\sigma^a)_{\alpha\beta}(\sigma^b)_{\gamma\delta} q^\alpha q^\beta q^\gamma q^\delta, \tag{2.8}$$

$$\tilde{P}^{ab}(3,1) = \frac{1}{4}(\sigma^a)_{\alpha\beta}(\sigma^b)_\gamma{}^\delta q^\alpha q^\beta q^\gamma p_\delta, \tag{2.9}$$

$$\tilde{P}^a(3,1) = \frac{1}{16}(\sigma^a)_{\alpha\beta}(q^\alpha q^\beta q^\gamma p_\gamma + q^{(\alpha} q^\beta p_\gamma q^{\gamma)} + q^{(\alpha} p_\gamma q^\beta q^{\gamma)} + p_\gamma q^\alpha q^\beta q^\gamma), \;\; etc. \tag{2.10}$$

By computing the algebra of these generators keeping only single commutator terms we obtain the classical higher spin algebra based on the Poisson bracket used in this context in the original work on conformal higher spins in three dimensions [6]. Instead, by quantizing the variables q^α, p_α and Weyl order them as above they generate the for us relevant higher algebra of $SO(3,2)$. Note that for all generators $G(2n)$ (which are of order $2n$ in q^α, p_α)

[2]A perhaps more appropriate definition of the Schouten tensor is used, e.g., in [12], which corresponds to the spin 3 field $\tilde{f}^{ab}$ in (2.5).

[3]The notation adopted by Vasiliev often use y^α which correspond to our q^α, p_α while the auxiliary z^α "coordinates" have no analogue here.

with n vector indices the ordering of the q and p operators do not matter and they are thus automatically ordered as required.

We give the operators q^α dimension $L^{\frac{1}{2}}$ and p_α dimension $L^{-\frac{1}{2}}$ which means that both A and F will be dimensionless. This will be useful later when we discuss how to construct S and T on the RHSs of (2.1) and (2.2). With these rules all multiplications can be viewed as *star products* which, however, has to be remembered since it is not explicitly shown by our notation.

We now turn to the LHS of (2.1). The derivative operator appearing there is just $\mathcal{D} = d + A$ where A is as defined above[4]. However, the scalar field $\Phi(x)$ is special and differs in its definition form A. Φ is expanded only in terms of the most special conformal generators $K^{a_1 \ldots a_n}$ which is the last one of the generators in each spin $s = n+1$ field A_n above and contains *only* the variables p_α, in fact, exactly $2n$ of them. We define the HS scalar field as follows

$$\Phi(x) = \Sigma_{n=0}^{\infty} (-i)^n \phi^{a_1 \ldots a_n}(x) K_{a_1 \ldots a_n}, \tag{2.11}$$

where the first term defines the usual scalar field $\phi(x)$ that will appear conformally coupled to spin 2 and all higher spin frame fields coming from A and with its own fifth order self-interaction in the Klein-Gordon equation.

The vacuum used in (2.1) is defined to be annihilated by the q^α operators making it translationally invariant in the sense that $P^a|0\rangle_q = 0$. Although Φ itself does not contain any q^α operators, the fact that A does will lead to the appearance of interaction terms already for spin 2. In particular, a correctly normalized $R\phi$ interaction term appears directly after starting the unfolding procedure as observed in [1]. The scalar field is conformal and thus of dimension $L^{-\frac{1}{2}}$ so the LHS of (2.1) is a one-form of dimension $L^{-\frac{1}{2}}$ which must be true also for S on the RHS of that equation. We will propose an expression for S below after explaining the structure of the second equation $F = T$.

The role of the vacuum in the unfolded equation (2.1) is clear and a well-known property of this kind of scalar field, see, e.g., [9]. However, one of the crucial points in this discussion is to understand the relation of the two field equations (2.1) and (2.2) where the former one involves the vacuum while the latter one does not and hence has components for every generator of the higher spin algebra. To make the following argument a bit more explicit we give the spin 2 and spin 3 equations coming from the generator decomposition of $F = 0$. Note, however, that the following spin 2 and 3 equations have been truncated to the single commutator terms for simplicity.

For spin 2 the equations are (in the gauge $b_\mu = 0$)[5]

$$F = 0: \quad T^a = 0, \tag{2.12}$$

$$R^a - 2\epsilon^a{}_{bc} e^b \wedge f^c = 0, \tag{2.13}$$

$$e^a \wedge f_a = 0, \tag{2.14}$$

$$D f^a = 0, \tag{2.15}$$

[4] The derivative D used frequently below is defined to contain only the spin 2 spin connection ω^a.

where $T^a = De^a = de^a + \omega^a{}_{bc}\omega^b \wedge e^c$ and $R^a = d\omega^a + \frac{1}{2}\omega^a{}_{bc}\omega^b \wedge \omega^c$. The second of these is useful for us since solving it for $f_\mu{}^a$ gives

$$f_{\mu\nu} = \frac{1}{2}(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R) = \frac{1}{2}S_{\mu\nu}, \tag{2.16}$$

where $S_{\mu\nu}$ is the Schouten tensor. The third equation in the above $F = 0$ spin 2 system then just says that the Schouten tensor is symmetric and the last equation that it satisfies the Cotton equation.

For spin 3 we give only the cascading equations (see the next section) used to express the spin 3 Schouten tensor $f_\mu{}^{ab}$ in terms of the frame field $e_\mu{}^{ab}$ (the full system including the constraint equations is discussed in [1, 16])

$$\begin{aligned}
&F^{ab}(4,0) = De^{ab} + e^c \wedge \tilde{e}^{d(a}\epsilon_{cd}{}^{b)} - (e^{(a} \wedge \tilde{e}^{b)} - trace) = 0,\\
&F^{ab}(3,1) = D\tilde{e}^{ab} - 2e^c \wedge \tilde{\omega}^{d(a}\epsilon_{cd}{}^{b)} - (e^{(a} \wedge \tilde{\omega}^{b)} - trace) - 4f^c \wedge e^{d(a}\epsilon_{cd}{}^{b)} = 0,\\
&F^{ab}(2,2) = D\tilde{\omega}^{ab} + 3e^c \wedge \tilde{f}^{d(a}\epsilon_{cd}{}^{b)} - (e^{(a} \wedge \tilde{f}^{b)} - f^{(a} \wedge \tilde{e}^{b)} - trace) + 3f^c \wedge \tilde{e}^{d(a}\epsilon_{cd}{}^{b)} = 0,\\
&F^{ab}(1,3) = D\tilde{f}^{ab} - 4e^c \wedge f^{d(a}\epsilon_{cd}{}^{b)} + (f^{(a} \wedge \tilde{\omega}^{b)} - trace) - 2f^c \wedge \tilde{\omega}^{d(a}\epsilon_{cd}{}^{b)} = 0,\\
&F^{ab}(0,4) = Df^{ab} + (f^{(a} \wedge \tilde{f}^{b)} - trace) + f^c \wedge \tilde{f}^{d(a}\epsilon_{cd}{}^{b)} = 0.
\end{aligned} \tag{2.17}$$

From the explicit structure of these equations (and the work in [1, 6]) it should be clear that they can all be solved algebraically except for the very last one which is the Cotton equation and that this works for all spins. The result of this procedure thus expresses the spin $s = n + 1$ Schouten tensor $f_\mu{}^{a_1 \ldots a_n}$ in terms of the frame field $e_\mu{}^{a_1 \ldots a_n}$, a relation that involves $2s - 2$ derivatives. In order to be able to introduce interactions, i.e., a stress tensor on the RHS of all the Cotton equations for arbitrary spin we must relax the equation $F = 0$ and instead consider $F = T$ where the RHS must have the property that all the field strengths $F^{a_1 \cdots a_n}(0, 2n)$ pick up the proper source terms.

Having concluded that all the component equations $F(n_q, n_p) = 0$ for $n_q > 0$ can be solved we note that the field strength F reduces to

$$F = \Sigma_{n=0}^{\infty}(-i)^n F^{a_1 \cdots a_n}(0, 2n)K_{a_1 \ldots a_n}, \tag{2.18}$$

i.e., it has become a field with the same structure as Φ defined above. This reduction of F may, however, not be compatible with the Bianchi identities. Also other parts of F that are assumed zero here are probably only so in the linear analysis performed in [1]. As will be elaborated upon elsewhere [16] some of the constraint equations of the spin 3 system along other generators than K^{ab} will contain the spin 2 Cotton tensor and will thus be affected by the introduction of source terms. Hence this description of the structure of F implies that F can not be made to act directly on the vacuum like in the unfolded equation (2.1) since then we would loose information. This suggests that the proper equation for F and T involving the vacuum is instead the integrability equation for (2.1) namely[5]

$$F\Phi|0\rangle_q = \mathcal{D}S|0\rangle_q, \tag{2.19}$$

[5] Note that applying another derivative gives an identity after using the Bianchi identity for F.

with F taking values in the whole HS algebra, which implies that

$$T\Phi|0\rangle_q = \mathcal{D}S|0\rangle_q, \tag{2.20}$$

from which it should be possible to construct T.

We have now explained how to view the two equations (2.1) and (2.2) and also defined the RHSs of these equations without giving any explicit expressions for them. The main result of this paper is that it is in fact possible to construct the RHSs producing a fully interacting theory with back reactions in both equations. We emphasize here that we have not yet provided a complete proof that the equations we propose constitute a consistent system. In this context it might also be relevant to analyze the equation (where $\star$ is the Hodge dual)

$$\mathcal{D} \star \mathcal{D}\Phi|0>= \mathcal{D} \star S|0>, \tag{2.21}$$

which follows directly from the unfolded equation and contains the Klein-Gordon equation already at level $n = 0$. It seems that for these different equations derived from the unfolded equation to be consistent with one another, the information of the original equation is merely reshuffled to different levels but otherwise the same. Under what conditions this is true (if at all) remains to be demonstrated, however.

Turning to the source terms we start by constructing S. It was explicitly shown in [1] that the unfolded equation with a zero RHS gives rise to the conformal interaction term $R\phi$ with the correct coefficient for three dimensions. The goal now is to construct a RHS such that also the fifth order interaction term is generated after unfolding the equation. In fact, also the scalar terms in the full Cotton equation (1.3) require the addition of a source term. The only structures that can be written down which are one-forms with dimension $L^{-\frac{1}{2}}$ and could generate the wanted terms are in fact,

$$S = -i\lambda_1 M(\Phi^*\Phi)\Phi - i\lambda_2 K(\Phi^*\Phi)^2\Phi, \tag{2.22}$$

where λ_1 and λ_2 are two free parameters. Here we have used the definitions

$$M = dx^\mu e_\mu{}^a M_a, \;\; K = dx^\mu e_\mu{}^a K_a, \tag{2.23}$$

where M_a and K_a are the spin 2 Lorentz and special conformal generators of dimension L^0 and L^1, respectively. Unfolding (2.1) indeed gives the correct Klein-Gordon equation at the spin 2 level (see below) and interestingly enough also the correct Cotton equation. As described for spin 3 in [1] this unfolding can be carried out further up in spin without any problems. There are, however, features involving infinite sets of higher spin terms in the full equations which probably means that the equations have to be iterated and truncated at some desired high spin level.

However, for this to work in the sense of producing the spin 2 Cotton equation with the correctly coupled scalar field as in (1.3) one further step is required. As we will clear below we have to make use of the possibility to shift the gauge fields in A by tensor terms which for spin 2 we choose as

$$\hat{A}_1 = A_1 + \lambda_1 M\phi^2. \tag{2.24}$$

We will, however, not work with this shifted gauge field but instead move the tensor term over to the RHS of the unfolded equation. Combining this term with the one already in S we find that the RHS becomes

$$S_M = -i\lambda_1 M(\phi^a\phi^b - 2\phi\phi^{ab})K_{ab} + ...)\Phi, \quad (2.25)$$

$$S_K = -i\lambda_2 K(\phi^4 + 2(\phi^a\phi^b - 2\phi\phi^{ab})\phi^2 K_{ab} + ...)\Phi. \quad (2.26)$$

Note that a corresponding term for the P_a generator does not exist since the term P in A is of dimension zero so it cannot contain any factors of Φ. It may be mentioned in this context that the unfolded equation will itself produce the full spin 2 Cotton equation with the stress tensor as a source. For higher spins one may speculate about the structure of the corresponding Cotton equations. For spin 3 for instance, the Cotton tensor is in the irrep **7** and has dimension L^{-4}, compared to L^{-3} for spin 2, and hence the two-scalar terms must contain one further derivative which should result from the unfolding. Also other more complicated terms are possible with derivatives distributed between scalars and higher spin frame fields in various ways. It is even possible that there is a non-derivative $e_{\mu ab}\phi^{10}$ term as a source in the spin 3 Cotton equation which may come from further terms in the source S. E.g., one may envisage terms containing $e_\mu{}^{ab}$ which must involve $e_\mu{}^{ab}M_{ab}$, $e_\mu{}^{ab}K_{ab}$, etc, multiplied by $|\Phi|^4\Phi$, $|\Phi|^8\Phi$, etc for dimensional reasons. How it is possible for such terms in S to affect the spin three equations will be clear below. Note that the issue of whether or not terms like these will contribute also to the Klein-Gordon equation depends on traces like $e^\mu{}_a e_\mu{}^{ab}$ being non-zero. However, this is probably not the case since they can be set to zero by higher spin "scale" transformations.

In a similar manner we may deduce the structure of T in the HS equation (2.2). T must be a two-form of zero dimension giving rise to, after unfolding, both $\partial_\mu\phi\partial_\nu\phi$ and $\phi D_\mu\partial_\nu\phi$ type terms. An especially intriguing fact is that derivative terms of the kind $\phi D_\mu\partial_\nu\phi$ may only arise through unfolding. The T that has these properties will not be presented here and we hope to come back to this question elsewhere. Note that a structure similar to S, i.e.,

$$T = -ig_1 \star P(\Phi^*\Phi) - ig_2 \star M(\Phi^*\Phi)^2 - ig_3 \star K(\Phi^*\Phi)^3, \quad (2.27)$$

where the two-forms are

$$\star P = \frac{1}{2} dx^\mu \wedge dx^\nu \epsilon_{\mu\nu}{}^\rho e_\rho{}^a P_a, \;\; etc. \quad (2.28)$$

will not suffice since terms with explicit derivatives seems to be needed. In fact, this follows directly from (2.20) which of course will imply relations between parameters in T and S. Nevertheless, it is the first term in (2.27) that has the correct structure to generate the required source term for the spin 2 Cotton equation in $F = T$. As for S also T will contain HS contributions of the kind $e_\mu{}^{ab}P_{ab}$ etc.

2.2 Explicit unfolding

In order to perform some checks we need to unfold the scalar equation

$$\mathcal{D}\Phi|0\rangle_q = S|0\rangle_q, \quad (2.29)$$

to find expressions for some of the first terms in the expansion. The point we want to emphasize here is that this equation contains, apart from the scalar field equation, also the higher spin field equations obtained by solving the equation $F = 0$ but now coupled to the scalar field. We now demonstrate this explicitly by deriving both the Klein-Gordon equation (1.1) and the spin 2 Cotton equation (1.2) from the unfolded equation (2.29).

We start by computing the first few levels of the left hand side of the unfolded equation. At level n the expressions multiplying $K_{a_1\ldots a_n}|0\rangle_q$ are (where $D = d + \omega(1,1)$ and $\mathcal{O}(HS)$ indicates further higher spin terms that can be computed when needed)

$$\begin{aligned}
\mathcal{D}\Phi|0\rangle_q: \quad & n=0: \quad (\partial_\mu\phi + \phi_\mu + \mathcal{O}(e_\mu{}^{ab}\phi_{ab}(s=3)+\ldots))|0\rangle_q, && (2.30)\\
& n=1: \quad (D_\mu\phi^a + f_\mu{}^a\phi + 6\phi_\mu{}^a + \mathcal{O}(s\geq 3))K_a|0\rangle_q, && (2.31)\\
& n=2: \quad (D_\mu\phi^{ab} + f_\mu{}^{(a}\phi^{b)} + 15\phi_\mu{}^{ab} + \mathcal{O}(s\geq 3))K_{ab}|0\rangle_q, && (2.32)
\end{aligned}$$

where we need to keep in mind that the uncontracted flat indices are always in irreps, i.e., in symmetrized traceless representations. This means that for levels $n \geq 1$ each equation splits into three irreducible parts $n-1$, n and $n+1$ obtained by multiplying it with the level one generators $P^\mu := e^\mu{}_a P^a$, $M^\mu := e^\mu{}_a M^a$ and $K^\mu := e^\mu{}_a K^a$, respectively. We refer to the resulting equations as n^-, n^0, n^+, respectively. Applying this procedure to the $n = 1$ equation above we find

$$\begin{aligned}
n=1^-: \quad & D_\mu\phi^\mu + f_\mu{}^\mu\phi + \mathcal{O}(s\geq 3), && (2.33)\\
n=1^0: \quad & \epsilon^{\mu\nu a}(D_\mu\phi_\nu + f_{\mu\nu}) + \mathcal{O}(s\geq 3), && (2.34)\\
n=1^+: \quad & D_{(\mu}\phi_{\nu)} + f_{(\mu\nu)} + 6\phi_{\mu\nu} + \mathcal{O}(s\geq 3). && (2.35)
\end{aligned}$$

Setting $\mathcal{D}\Phi|0\rangle_q = 0$ we can insert the level $n = 0$ result into these equations and find

$$\begin{aligned}
n=1^-: \quad & -\Box\phi + f_\mu{}^\mu\phi + \mathcal{O}(s\geq 3) = 0, && (2.36)\\
n=1^0: \quad & \epsilon^{\mu\nu a} f_{\mu\nu} + \mathcal{O}(s\geq 3) = 0, && (2.37)\\
n=1^+: \quad & -D_{(\mu}\partial_{\nu)}\phi + f_{(\mu\nu)} + 6\phi_{\mu\nu} + \mathcal{O}(s\geq 3) = 0. && (2.38)
\end{aligned}$$

At level 2 we find the following LHSs of the unfolded equation

$$\begin{aligned}
n=2^-: \quad & D_\mu\phi^{\mu a} + \frac{1}{2}f_\mu{}^\mu\phi^a + \frac{1}{6}f^{ab}\phi_b + \mathcal{O}(s\geq 3), && (2.39)\\
n=2^0: \quad & \epsilon^{\mu\nu(a}D_\mu\phi_\nu{}^{b)} + \frac{1}{2}\epsilon^{\mu\nu(a}f_\mu{}^{b)}\phi_\nu + \mathcal{O}(s\geq 3), && (2.40)\\
n=2^+: \quad & D_{(\mu}\phi_{\nu\rho)} + f_{(\mu\nu}\phi_{\rho)} + 15\phi_{\mu\nu\rho} + \mathcal{O}(s\geq 3). && (2.41)
\end{aligned}$$

To get a feeling for the non-trivial information in these equations we again assume $\mathcal{D}\Phi|0\rangle_q = 0$ and continue by analyzing the first of the $n = 2$ equations, the 2^- one. To do that we need to use information from the two lower levels. This gives (dropping $s \geq 3$ terms)

$$\frac{1}{6}D_\nu(D^{(\nu}\partial^{\mu)}\phi - \frac{1}{3}g^{\nu\mu}\Box\phi - f^{(\nu\mu)}\phi + \frac{1}{3}g^{\nu\mu}f_\rho{}^\rho\phi) - \frac{1}{2}f_\nu{}^\nu D^\mu\phi - \frac{1}{6}f^{\mu\nu}\partial_\nu\phi = 0, \qquad (2.42)$$

which simplifies to

$$\Box\partial_\mu\phi - \frac{1}{3}D_\mu\Box\phi - (D_\nu f_\mu{}^\nu)\phi - 2f_\mu{}^\nu\partial_\nu\phi + \frac{1}{3}(D_\mu f_\nu{}^\nu)\phi - \frac{8}{3}f_\nu{}^\nu\partial_\mu\phi = 0. \tag{2.43}$$

Then using the fact (where the zero torsion condition is assumed)

$$\Box\partial_\mu\phi = D_\mu\Box\phi + R_\mu{}^\nu\partial_\nu\phi \tag{2.44}$$

and the Klein-Gordon equation, we find the above 2^- equation to read

$$(R_{\mu\nu} - 2f_{\mu\nu} - 2f_\rho{}^\rho g_{\mu\nu})\partial^\nu\phi - (D_\nu f_\mu{}^\nu - D_\mu f_\nu{}^\nu)\phi = 0. \tag{2.45}$$

We note then that this equation becomes an identity if we set

$$f_{\mu\nu} = \frac{1}{2}S_{\mu\nu} = \frac{1}{2}(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R), \tag{2.46}$$

where $S_{\mu\nu}$ is the Schouten tensor. As we have seen above in (2.16), the equation $F = 0$ also contains this information.

The equation $n = 1^+$ above can now be seen to play an interesting role in rewriting the Cotton equation in a way that will help us to guess source terms for the entire higher spin system. The point is that if we use (2.46) in the $n = 1^+$ equation we can eliminate the term $D_{(\mu}\phi_{\nu)} = -D_\mu\partial_\nu\phi$ from the Cotton equation in (1.3). This gives

$$C_{\mu\nu} = \frac{3g}{8}(\phi_\mu\phi_\nu - 2\phi\phi_{\mu\nu})|_{\mathbf{5}}, \tag{2.47}$$

which is the form of the Cotton equation we will now show can be derived from the unfolded equation in (2.29).

Now we turn to the second of the $n = 2$ level equations, the 2^0 in the irrep $\mathbf{5}$. Making use of the lower level equations it becomes

$$\epsilon^{\mu\nu(a}(D_\mu D_\nu\partial^{b)}\phi - (D_\mu f_\nu{}^{b)})\phi - 2f_\mu{}^{b)}\partial_\nu\phi) = 0. \tag{2.48}$$

Using the Ricci identity this equation reads

$$\epsilon^{\mu\nu(a}(R_\mu{}^{b)}\partial_\nu\phi - (D_\mu f_\nu{}^{b)})\phi - 2f_\mu{}^{b)}\partial_\nu\phi) = 0, \tag{2.49}$$

and setting $f_{\mu\nu} = \frac{1}{2}S_{\mu\nu}$ as found above it reduces to

$$-\frac{1}{2}C^{ab}\phi = 0, \tag{2.50}$$

where $C^{ab} = \epsilon^{\mu\nu(a}D_\mu R_\nu{}^{b)}$ is the Cotton tensor. Again we find information present also in the equation $F = 0$. Thus is it clear that while the $F = 0$ equations contain, of course, only higher spin dynamics without scalar field sources the unfolded equation $\mathcal{D}\Phi|0\rangle_q = 0$ contains dynamical information for both the scalar field and the higher spin fields but without any non-trivial couplings between the scalar field and the higher spin ones. Introducing sources must thus be done for both equations in a consistent way. We will address this issue again below.

We now introduce the non-zero source terms (2.25):

$$\mathcal{D}\Phi|0\rangle_q = (S_M + S_K)|0\rangle_q. \tag{2.51}$$

One crucial property of this expression for the source is that it is zero at level $n = 0$, and that S_M contributes only to the n^0 equations at level n while S_K contributes only to the n^- equations. Thus there are no source terms affecting the n^+ equations which therefore are the same as for $\mathcal{D}\Phi|0\rangle_q = 0$ where it is used to determine the fields $\phi^{a_1\ldots a_{n+1}}$ in terms of fields at lower levels. This is seen as follows: consider a general term at level n in the expansion of $K_\mu\Phi^5|0\rangle_q = e_\mu{}^a K_a\Phi^5|0\rangle_q$ which we write as $e_\mu{}^a(\Phi^5)^{b_1\ldots b_n}K_{ab_1\ldots b_{n-1}}|0\rangle_q$. Contracting it with P^μ then gives $K_{b_1\ldots b_{n-1}}$. Contraction with M^μ gives instead $e_\mu{}^a\epsilon^\mu{}_{(a}{}^c K_{b_2\ldots b_{n-1})c} = 0$ and using K^μ one gets $e^{\mu a}K_{\mu ab_1\ldots b_{n-1}} = 0$. S_M works in a similar way with contributions only to the n^0 equations.

We will now continue to analyze the effects of adding the explicit source terms given in (2.25):

$$S_M = -i\lambda_1 M(\phi^a\phi^b - 2\phi\phi^{ab})K_{ab} + \ldots)\Phi, \tag{2.52}$$

$$S_K = -i\lambda_2 K(\phi^4 + 2(\phi^a\phi^b - 2\phi\phi^{ab})\phi^2 K_{ab} + \ldots..)\Phi. \tag{2.53}$$

Using the results

$$[M_a, K_{bc}] = -2i\epsilon_{a(b}{}^d K_{c)d}, \tag{2.54}$$

$$M^a[M_a, K_{bc}]|0\rangle_q = -12K_{bc}|0\rangle_q, \tag{2.55}$$

the RHS of the unfolded equation is at the first few levels

$$M^\mu(S_M)_\mu|0\rangle_q: \quad = 0, \quad n = 0, \tag{2.56}$$

$$= 0, \quad n = 1^0, \tag{2.57}$$

$$= -12\lambda_1(\phi^a\phi^b - 2\phi\phi^{ab})\phi K_{ab}|0\rangle_q, \quad n = 2^0. \tag{2.58}$$

In the case of S_K we need the results

$$[P_a, K_b] = -2i\epsilon_{ab}{}^c M_c - 2i\eta_{ab}D \Rightarrow [P^a, K_a]|0\rangle_q = -3|0\rangle_q, \tag{2.59}$$

$$[P_a, K_{bc}]|0\rangle_q = -6(\eta_{a(b}K_{c)} - \tfrac{1}{3}\eta_{bc}K_a)|0\rangle_q \Rightarrow [P^a, K_{ab}]|0\rangle_q = -10K_b|0\rangle_q. \tag{2.60}$$

We find the following contributions to the n^- equations

$$P^a 6\lambda K_a\Phi^5|0\rangle_q \;= 0, \quad n = 0, \tag{2.61}$$

$$= -3\lambda_2\phi^5|0\rangle_q, \quad n = 1^-, \tag{2.62}$$

$$= -10\lambda_2\phi^b\phi^4 K_b|0\rangle_q, \quad n = 2^-. \tag{2.63}$$

$$\tag{2.64}$$

With these results for the source terms of the unfolded equation

$$\mathcal{D}\Phi|0\rangle_q = (S_M + S_K)|0\rangle_q, \tag{2.65}$$

the first few level equations become, dropping terms involving higher spin fields,

$$\begin{aligned}
n = 0: &\quad \phi_\mu = -\partial_\mu \phi, && (2.66)\\
n = 1^-: &\quad \Box\phi - \tfrac{1}{8}R\phi - 3\lambda_2\phi^5 = 0, && (2.67)\\
n = 1^0: &\quad 0 = 0, && (2.68)\\
n = 1^+: &\quad 6\phi_{\mu\nu} = (D_{(\mu}D_{\nu)}\phi - f_{\mu\nu}\phi)|_{\mathbf{5}}, && (2.69)\\
n = 2^-: &\quad (R_{\mu\nu} - 2f_{\mu\nu} - 2f_\rho{}^\rho g_{\mu\nu})\partial^\nu\phi - (D_\nu f_\mu{}^\nu - D_\mu f_\nu{}^\nu)\phi = 0, && (2.70)\\
n = 2^0: &\quad C_{\mu\nu} = 36\lambda_1(\phi_\mu\phi_\nu - 2\phi\phi_{\mu\nu})|_{\mathbf{5}}, && (2.71)\\
n = 2^+: &\quad \phi_{\mu\nu\rho} = -\tfrac{1}{15}(D_{(\mu}\phi_{\nu\rho)} + f_{(\mu\nu}\phi_{\rho)})|_{\mathbf{7}}. && (2.72)
\end{aligned}$$

where we have adopted the solution $f_{\mu\nu} = \frac{1}{2}S_{\mu\nu} = \frac{1}{2}(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R)$ to the $n = 2^-$ equation in order to obtain the Cotton equation from the $n = 2^0$ equation. The Cotton equation obtained this way has the same structure as the one we were seeking namely the one given in (2.47) so this seems to be on the right track. A crucial further test is to construct the source T in the adjoint equation $F = T$ such that the same Cotton equation results as discussed above after equation (2.27). This should be possible to do and we hope to come back to this in a future publication.

We should also emphasize another feature of the calculation leading to these results. The fact that the scalar self-interaction term $K|\Phi|^4\Phi$ gives rise to new terms in both the 1^- and 2^- equations leads to a consistency check in the sense that hese terms are seen to cancel in the the 2^- equation and hence the result quoted for $f_{\mu\nu}$ is not affected by the addition of the $K|\Phi|^4\Phi$ term.

3 A cascading Lagrangian

In this section we will make use of a feature of the Chern-Simons gauge theory for the higher spin gauge field A that allows us to show that the component Lagrangian is naturally expressed in terms of the generalized spin connections as suggested in [1]. We will demonstrate explicitly that it is possible to derive such a Lagrangian once a certain subset of the $F = 0$ component equations are solved. This subset of equations, which will be called cascading, does not include the spin s Cotton equations which are therefore obtained by varying the resulting Lagrangian with respect to the frame fields for each spin $s \geq 2$. $F = 0$ contains a number of other equations that one must use to determine other fields contained in A or prove are identities; for a complete discussion of the spin three situation see [1].

We use here the results coming from the single commutator terms in the spin 2 and 3 cases as examples of the technique. However, these examples make it plausible that this procedure works for all spins and in the full star product formulation where all multi-commutators and non-linearities are included. Its origin is in the gauge Chern-Simons theory

$$S = \frac{1}{2}Tr\int(AdA + \frac{2}{3}A^3), \tag{3.1}$$

where the trace is in the higher spin algebra which should generate precisely the terms in the Lagrangian used in the cascading procedure described below. This Lagrangian can in

principle be written out explicitly in terms of all the fields appearing in A [7]. This will produce very complicated expressions and seems useful only for lower spin truncations. To get from this first order formulation to a "second" order one in terms of only the frame fields seems even more complicated and any kind of simplifications that can be utilized in this context would be welcome. Below we will describe one potentially useful feature of this kind.

The standard spin 2 Chern-Simons like action reads in terms of $\omega_1 := \omega(1,1)$

$$S_2 = \frac{1}{2}\int(\omega_1 d\omega_1 + \frac{2}{3}\omega_1 \wedge \omega_1 \wedge \omega_1), \tag{3.2}$$

which leads to the field equation $C_{\mu\nu} = 0$ for the Cotton tensor $C_{\mu\nu} = \epsilon_\mu{}^{\alpha\beta} D_\alpha(R_{\beta\nu} - \frac{1}{4}g_{\beta\nu}R)$. Here we use the notation from [1] for the $s = 2$ spin connection obtained by solving the zero torsion condition. In that paper it was suggested that this spin 2 action has a generalization to arbitrary spin in the sense that the action is naturally expressed in terms of $\tilde{\omega}_n := \tilde{\omega}(n,n)$ which is a one-form in the irrep n of $SO(1,2)$. (A spin 3 example is $\tilde{\omega}^{ab}$ appearing in the expansion of A_2 in (2.5).)

Here we will show how to derive the action for spin 3 which is of the suggested form

$$S_3 = \frac{1}{2}\int \tilde{\omega}_2 D\tilde{\omega}_2 = \frac{1}{2}\int(\tilde{\omega}_2 d\tilde{\omega}_2 + \tilde{\omega}_2 \wedge \omega_1 \wedge \tilde{\omega}_2). \tag{3.3}$$

Note that a cubic term with three $\tilde{\omega}_2$ does not exist. However, once the multi-commutators are taken into account there will appear new interaction terms containing spin connections with arbitrarily high spin. Also other higher spin fields will occur in these interaction terms and to be clear about which fields we talk about we consider again the spin 3 higher spin gauge field in (2.5)

$$A_2 = e^{ab}P_{ab} + \tilde{e}^{ab}\tilde{P}_{ab} + \tilde{e}^a\tilde{P}_a + \tilde{\omega}^{ab}\tilde{M}_{ab} + \tilde{\omega}^b\tilde{M}_a + \tilde{b}\,\tilde{D} + \tilde{f}^a\tilde{K}_a + \tilde{f}^{ab}\tilde{K}_{ab} + f^{ab}K_{ab}. \tag{3.4}$$

Solving the first four of the equations in (2.17) will result in a cascading sequence of relations that will express the one-form field f^{ab}, called the spin 3 Schouten tensor, in terms of the frame field e^{ab} each step producing a new derivative. The last equation in (2.17) is then the spin 3 Cotton equation containing five derivatives. The action we derive here uses only the field $\tilde{\omega}^{ab}$ in A_2 denoted $\tilde{\omega}_2$ in (3.3). In fact, as we will see below also the field $\tilde{b}$ in (2.5) will appear in the action but it turns out that this field is expressible in terms of $\tilde{\omega}_2$ as shown in [1].

The main goal of this section is to give a simple procedure for deriving a Lagrangian that gives the full non-linear Cotton equation for any spin, and indeed for the whole higher spin system. This result follows provided some basic conditions to be specified below are met. Here we give the main ideas and the details only for spin 2 and 3 but it is likely that this method can be generalized to the whole higher spin theory.

We will now show how one can derive an action that automatically gives rise to the fifth order Cotton equation for this spin 3 system. In order to streamline the discussion we simplify the spin 3 equations in (2.17) as far as possible without destroying features of the system that are relevant for this particular discussion. First we note that the spin 3

equations contain fields from the spin 2 system that we can discard at this point but put back if a complete analysis is required. This statement applies to all terms containing the spin 2 fields ω^a and f^a but not to the terms with a dreibein e^a. This reduces the equations to

$$de^{ab} + e^c \wedge \tilde{e}^{d(a}\epsilon_{cd}{}^{b)} - (e^{(a} \wedge \tilde{e}^{b)} - trace) = 0, \tag{3.5}$$
$$d\tilde{e}^{ab} - 2e^c \wedge \tilde{\omega}^{d(a}\epsilon_{cd}{}^{b)} - (e^{(a} \wedge \tilde{\omega}^{b)} - trace) = 0, \tag{3.6}$$
$$d\tilde{\omega}^{ab} + 3e^c \wedge \tilde{f}^{d(a}\epsilon_{cd}{}^{b)} - (e^{(a} \wedge \tilde{f}^{b)} - trace) = 0, \tag{3.7}$$
$$d\tilde{f}^{ab} - 4e^c \wedge f^{d(a}\epsilon_{cd}{}^{b)} = 0, \tag{3.8}$$
$$df^{ab} = 0. \tag{3.9}$$

We now need to make use of the possibility to gauge fix the higher spin symmetries to further simplify these equations. As explained in [1] the field $\tilde{e}^a$ can be set to zero by using the symmetries related to the parameters $\tilde{\Lambda}^{ab}(2,2)$, $\tilde{\Lambda}^a(2,2)$, and $\tilde{\Lambda}(2,2)$. This sets to zero the last term in the first equation above. In order to eliminate also the last term in the second and third equations we need to be able to choose a gauge where $\tilde{\omega}^a = e^a\hat{\omega}$ and $\tilde{f}^a = e^a\hat{f}$. However, while this is possible for $\tilde{\omega}^a$ it is not so for $\tilde{f}^a$. In the former case we can use $\tilde{\Lambda}^{ab}(1,3)$ and $\tilde{\Lambda}^a(1,3)$ to establish this fact but in the latter case we have only $\tilde{\Lambda}^{ab}(0,4)$ at our disposal which means that the best we can do is to gauge fix to

$$\tilde{f}_\mu{}^a = \epsilon_\mu{}^{ab}\hat{f}_b + e_\mu{}^a\hat{f}, \tag{3.10}$$

which unfortunately will complicate the situation somewhat.

Instead of trying to construct a Lagrangian directly for the frame fields and then perform a variation with respect to the frame fields to obtain the Cotton equations these can be obtained in a manner that is slightly easier if we make use of the description of this system as a Chern-Simons gauge theory for the conformal group $SO(3,2)$. In order to see how this is done we consider first the spin 2 Chern-Simons system which is given in terms of the gauge field

$$A_1 = e^aP_a + \omega^aM_a + bD + f^aK_a, \tag{3.11}$$

where the $SO(3,2)$ generators P_a, M_a, D, K_a have been assigned one gauge field each. The exercise is then to solve the zero field strength equation $F_1 = 0$ which if decomposed along the different generators become (here the Riemann tensor is $R^a = d\omega^a + \frac{1}{2}\epsilon^a{}_{bc}\omega^b \wedge \omega^c$ and we have imposed the gauge $b = 0$)

$$\begin{aligned} &T^a = de^a + \epsilon^a{}_{bc}\omega^b \wedge e^c = 0,\\ &R^a - 2\epsilon^a{}_{bc}e^b \wedge f^c = 0,\\ &Df^a = 0, \end{aligned} \tag{3.12}$$

which we call the cascading equations while the remaining equation

$$e^a \wedge f_a = 0, \tag{3.13}$$

is a constraint on the solution of the cascading system above. The first equation is solved for the spin connection in terms of the dreibein, the second for f^a in terms of the Riemann

tensor with the result that $f^a_\mu e_{\nu a}$ is just the symmetric Schouten tensor. The last equation is then a constraint that is automatically satisfied while the last of the cascading equations becomes the Cotton equation. The goal is now to use these cascading equations to show that the variation of the action gives the Cotton equation.

We start the cascading procedure from

$$L_1 = -2e^a \wedge Df_a. \tag{3.14}$$

The variation of L_1 is

$$\delta L_1 = -2\delta e^a \wedge Df_a - 2e^a \wedge D\delta f_a - 2e^a \wedge \epsilon_{abc}\delta\omega^b \wedge f^c, \tag{3.15}$$

which would give the Cotton equation by demanding $\delta L_1 = 0$ if the last two terms could be gotten rid off. To achieve this we note first that the second term vanishes due to the torsion constraint after an integration by parts. To deal with the last term we add the standard Chern-Simons term

$$L_2 = \tfrac{1}{2}\omega^a \wedge d\omega_a + \tfrac{1}{6}\epsilon^{abc}\omega_a \wedge \omega_b \wedge \omega_c, \tag{3.16}$$

whose variation is

$$\delta L_2 = \delta\omega_a \wedge R^a = 2\delta\omega^a \wedge \epsilon_{abc}e^b \wedge f^c, \tag{3.17}$$

where we have used the second equation in (3.12) in the last equality. Thus we obtain the Cotton equation as a result of varying the Lagrangian $L = L_1 + L_2$. However, $L_1 = 0$ after an integration by parts as a consequence of the torsion constraint which is assumed solved in this analysis. This implies that the Lagrangian L_2 alone provides the Cotton equation when varied with respect to the dreibein ${e_\mu}^a$. This derivation of the Cotton equation is a bit too trivial to be interesting but for spin 3 and higher it seems to simplify the calculation of the spin s Cotton equation quite a bit. Recall that these equations are of order $2s-1$ in derivatives.

We now turn to the spin 3 system and repeat these steps. To this end we note that the spin 3 Cotton equation in the simplified version given above follows trivially by varying the Lagrangian

$$L_1 = e^{ab} \wedge df_{ab}, \tag{3.18}$$

with respect to the explicit frame field. However, this conclusion is only correct if we can eliminate the second term in its variation

$$\delta L_1 = \delta e^{ab} \wedge df_{ab} + e^{ab} \wedge d\delta f_{ab}. \tag{3.19}$$

But this can be done by adding another term to the Lagrangian whose variation cancels the last unwanted term. The term we need to add is

$$L_2 = e^c \wedge \tilde{e}^{da}{\epsilon_{cd}}^b \wedge f_{ab}. \tag{3.20}$$

The reason this works is that in the variation

$$\delta L_2 = e^c \wedge \tilde{e}^{da}{\epsilon_{cd}}^b \wedge \delta f_{ab} + e^c \wedge \delta\tilde{e}^{da}{\epsilon_{cd}}^b \wedge f_{ab}, \tag{3.21}$$

the first term equals $-de^{ab} \wedge \delta f_{ab}$ by using the field equation for e^{ab} coming from $F = 0$ (recall that we are in a gauge where $\tilde{e}^a = 0$). To make use of this field equation is of course allowed here since it is algebraic and actually solved so that it is identically satisfied. This fact can now be used for all the "field equations" in $F = 0$ except the last one which is the five derivative Cotton equation.

Having established this cancelation we now need to add a further term to cancel also the second term in δL_2 above. The required term is

$$L_3 = -\frac{1}{4}\tilde{e}^{ab} \wedge d\tilde{f}_{ab}, \tag{3.22}$$

whose variation can be written, again making use of the algebraic "field equations" this time the ones for $\tilde{f}^{ab}$ and $\tilde{e}^{ab}$, as

$$\delta L_3 = -\frac{1}{4}\delta\tilde{e}^{ab} \wedge d\tilde{f}_{ab} - \frac{1}{4}\tilde{e}^{ab} \wedge d\delta\tilde{f}_{ab} = \delta\tilde{e}^{ad} \wedge e^c \wedge f_{ab}\epsilon_{cd}{}^b - \frac{1}{2}e^c \wedge \tilde{\omega}^{da}\epsilon_{cd}{}^b \wedge \delta\tilde{f}_{ab}. \tag{3.23}$$

The remaining term is then the last one in the previous equation which we cancel by adding

$$L_4 = \frac{1}{2}e^c \wedge \tilde{\omega}^{da} \wedge \tilde{f}_{ab}\epsilon_{cd}{}^b, \tag{3.24}$$

which varies into

$$\delta L_4 = -\frac{1}{2}e^c \wedge \delta\tilde{\omega}_{ab} \wedge \tilde{f}^{da}\epsilon_{cd}{}^b + \frac{1}{2}e^c \wedge \tilde{\omega}^{da} \wedge \delta\tilde{f}_{ab}\epsilon_{cd}{}^b. \tag{3.25}$$

After canceling the last term against the same term coming from δL_3 we are left with the first term in δL_4 which we write as

$$-\frac{1}{6}\delta\tilde{\omega}^{ab} \wedge (d\tilde{\omega}_{ab} - e_a \wedge \tilde{f}_b). \tag{3.26}$$

The first term in this expression is canceled by the variation of

$$L_5 = \frac{1}{12}\tilde{\omega}^{ab} d\tilde{\omega}_{ab}. \tag{3.27}$$

Now we can use the algebraic equations from $F = 0$ again to find that $L_1 + L_2 = 0$ and $L_3 + L_4 = 0$. Since the procedure stops here L_5 is actually (the main part of) the final answer and is precisely the Lagrangian proposed in [1].

We have, however, still one term that we need to deal with, namely the second term in (3.26), which as we will now see is of a slightly different nature. We start by adding

$$L_6 = -\frac{1}{6}\tilde{\omega}^{ab} \wedge e_a \wedge \tilde{f}_b. \tag{3.28}$$

Varying $\tilde{\omega}$ gives the term we need to cancel in (3.26) leaving us with the term

$$-\frac{1}{6}\tilde{\omega}^{ab} \wedge e_a \wedge \delta\tilde{f}_b. \tag{3.29}$$

Now we must consult [1] where it was shown that the one-forms $\tilde{f}^a$ and $\tilde{b}$ are given by

$$\tilde{f}_\mu{}^a = \tilde{f}_{[\mu\nu]}e^{\nu a} + e_\mu{}^a\hat{f} = -\frac{3}{8}\partial_{[\mu}\tilde{b}_{\nu]} + e_\mu{}^a\hat{f}, \quad \tilde{b}_\mu = \frac{1}{4}\tilde{\omega}_{\nu\mu}{}^\nu. \tag{3.30}$$

In the present analysis where we keep track of only the spin 3 fields, the relevant expression we need is $\delta\tilde{f}_{[\mu\nu]} = -\frac{3}{8}\partial_{[\mu}\delta\tilde{b}_{\nu]}$ which implies that we can rewrite (3.29) as

$$-\frac{1}{6}\tilde{\omega}^{ab}\wedge e_a\wedge\delta\tilde{f}_b = \frac{1}{8}\tilde{b}\wedge d\delta\tilde{b}. \tag{3.31}$$

Thus the last spin 3 term to add is

$$L_7 = -\frac{1}{16}\tilde{b}\wedge d\tilde{b}. \tag{3.32}$$

We have therefore shown that the spin 3 part of the Lagrangian reads

$$L = L_5 + L_6 + L_7 = \frac{1}{12}\tilde{\omega}^{ab}\wedge d\tilde{\omega}_{ab} - \frac{1}{6}\tilde{\omega}^{ab}\wedge e_a\wedge\tilde{f}_b - \frac{1}{16}\tilde{b}\wedge d\tilde{b}. \tag{3.33}$$

The second term on the RHS is related (see above) to the last one and we find the final form of the Lagrangian to be

$$L = \frac{1}{12}\tilde{\omega}^{ab}\wedge d\tilde{\omega}_{ab} + \frac{1}{16}\tilde{b}\wedge d\tilde{b}, \tag{3.34}$$

which is therefore expressed entirely in terms of the spin connection $\tilde{\omega}^{ab}$ of the spin 3 sector.

One also needs to verify that the remaining constraint equations are satisfied as explained in [1]. It would then be interesting to see how the different steps in the cascading procedure are affected by increasing the spin, adding non-linear terms and coupling the system to other fields. For the spin 2 - spin 3 system these questions may be answered by the analysis of the full non-linear equations in [16]. As noted previously in this section the cascading trick is suggested by writing out the original Chern-Simons action for A using the trace over the higher spin algebra at each spin level separately as done in [7].

4 Conclusions

In this paper we have continued the approach to conformal higher spin theories in three dimensions set up recently in [1]. There it was emphasized that the unfolded equation $\mathcal{D}\Phi|0\rangle_q = 0$ for the higher spin algebra based on $SO(3,2)$, the conformal group in three dimensional space-time, realized in terms of two hermitian spinor operators q^α, p_α satisfying $[q^\alpha, p_\beta] = i\delta^\alpha_\beta$, produces the correct Klein-Gordon equation for a conformal scalar coupled to the spin 2 metric and its generalization to spin 3[6].

Here we take this approach some steps further by proposing an unfolded equation for a scalar field coupled to all higher spins ≥ 2 including the ϕ^5 self-interaction term in the Klein-Gordon equation. The expected scalar interactions with the spin 2 and higher spin fields show also up in the field equations and are produced in the unfolding. That the correct spin 2 Cotton equation is obtained is checked explicitly while for spin 3 the corresponding equation can easily be derived from this setup but it needs to be checked independently. Such a check would strengthen the argumentation for the higher spin equations suggested here.

[6] Note that there are other spin 3 terms in the Klein-Gordon equation than the one presented in [1].

We also present a simple method by which the Lagrangian for each higher spin field can be derived starting from the $F = 0$ equation valued in the higher spin algebra. This is demonstrated in section 3 for a truncated version of the spin 2 and 3 equations and can be seen to be a consequence of expanding the Chern-Simons gauge theory action using the trace over the entire higher spin algebra. Then a cascading trick leaves the whole action written in a form where the spin connections for each spin play a central role. The spin connections are here expressed as $s-1$ derivatives acting on the spin s frame field implying that $\mathcal{L}_n = \omega_n d\omega_n$ for $s = n+1$ contains $2s-1$ derivatives as it should. The result is a "second" order formalism type Chern-Simons Lagrangian generalizing the standard one $\mathcal{L}_1(\omega_1(e))$ for spin 2 to all spins. The cascading is here only performed for the linearized theory but will most likely give the full answer once all interaction terms are included.

This approach also suggests a way to write down a higher spin Lagrangian in the higher spin language. One may try to combine the gauge Chern-Simons action $S = \frac{1}{2}Tr\int(AdA + \frac{2}{3}A^3)$ discussed in the previous paragraph (and in section 3) with other expressions like (the star $\star$ is here a Hodge dual while the star product is implicit)

$$S = \int {}_p\langle 0|\tilde{\Phi}^*(q)\mathcal{D}\star\mathcal{D}\Phi(p)|0\rangle_q, \tag{4.1}$$

where we have introduced a dual scalar field $\tilde{\Phi}(q)$ which is expanded in terms of even powers of q^α instead of p_α as for the ordinary scalar field $\Phi(p)$. By assumption the dual field $\tilde{\Phi}(q) = \phi(x) + \tilde{\phi}_a(x)P^a + \ldots$ where $P^a = -\frac{1}{2}(\sigma^a)_{\alpha\beta}q^\alpha q^\beta$ is non-zero on the vacuum ${}_p\langle 0|$ which is used to produce a well-defined inner product ${}_p\langle 0|0\rangle_q = 1$.

This action can be expanded in component fields whose field equations should correspond to the scalar field equation $\mathcal{D}\star\mathcal{D}\Phi(p)|0\rangle_q = 0$. The source S can probably be hidden in the covariant derivative. The unfolded equation $\mathcal{D}\Phi(p)|0\rangle_q = 0$ could then be regarded as a solution to this equation which means that some information is lost if one instead solves only $\mathcal{D}\star\mathcal{D}\Phi(p)|0\rangle_q = 0$. It would be nice to have an action principle that directly generates the unfolded equation as the field equation. Interaction terms for the scalar field may also arise by considering actions of the kind

$$S_6 = \int {}_p\langle 0|(\tilde{\Phi}^*(q))^m \star (\Phi(p))^n|0\rangle_q|_{m+n=6}. \tag{4.2}$$

The actions considered here are dimensionless and the integrands are three-forms but they seem not to produce in a simple way the field equations used previously in this paper. The main reason for this is that although the dual field $\tilde{\Phi}(q)$ is here assumed to start with $\phi(x)$ the terms of higher order in q^α will probably be very complicated (even non-local).

Another potentially interesting aspect arises if this higher spin theory can be generalized to contain the topologically gauged spin theories derived in [2, 3, 17]. Then perhaps the background solutions found in [17, 18] could be lifted to the higher spin theory which could then provide information about how to write, e.g., an action also in AdS_3, Schroedinger and the semi-flat Schroedinger geometries discussed in [18, 19]. If this turns out to work it would give additional support for a "sequential AdS/CFT" phenomenon as suggested in [20] where Neumann boundary conditions and the associated dynamical conformal boundary theories play a crucial role.

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